Volume: The Shell Method

- Find the volume of a solid of revolution using the shell method.
- Compare the uses of the disk method and the shell method.

The Shell Method

In this section, you will study an alternative method for finding the volume of a solid of revolution. This method is called the shell method because it uses cylindrical shells. A comparison of the advantages of the disk and shell methods is given later in this section.

To begin, consider a representative rectangle as shown in Figure 7.27, where is the width of the rectangle, is the height of the rectangle, and is the distance between the axis of revolution and the center of the rectangle. When this rectangle is revolved about its axis of revolution, it forms a cylindrical shell (or tube) of thickness .

To find the volume of this shell, consider two cylinders. The radius of the larger cylinder corresponds to the outer radius of the shell, and the radius of the smaller cylinder corresponds to the inner radius of the shell. Because is the average radius of the shell, you know the outer radius is and the inner radius is .

So, the volume of the shell is

\[
\text{Volume of shell} = (\text{volume of cylinder}) - (\text{volume of hole})
\]

\[
= \pi \left( p + \frac{w}{2} \right)^2 h - \pi \left( p - \frac{w}{2} \right)^2 h
\]

\[
= 2\pi phw
\]

\[
= 2\pi(p)(\text{average radius})(\text{height})(\text{thickness}).
\]

You can use this formula to find the volume of a solid of revolution. Assume that the plane region in Figure 7.28 is revolved about a line to form the indicated solid. If you consider a horizontal rectangle of width then, as the plane region is revolved about a line parallel to the -axis, the rectangle generates a representative shell whose volume is

\[
\Delta V = 2\pi[p(y)h(y)] \Delta y.
\]

You can approximate the volume of the solid by such shells of thickness \(\Delta y\), height \(h(y)\), and average radius \(p(y)\).

\[
\text{Volume of solid} = \sum_{i=1}^{n} 2\pi \left[ p(y_i)h(y_i) \right] \Delta y = 2\pi \sum_{i=1}^{n} \left[ p(y_i)h(y_i) \right] \Delta y
\]

This approximation appears to become better and better as \(\Delta \to 0\) \((n \to \infty)\). So, the volume of the solid is

\[
\text{Volume of solid} = \lim_{|\Delta|\to 0} 2\pi \sum_{i=1}^{n} \left[ p(y_i)h(y_i) \right] \Delta y
\]

\[
= 2\pi \int_{c}^{d} \left[ p(y)h(y) \right] dy.
\]
EXAMPLE 1 Using the Shell Method to Find Volume

Find the volume of the solid of revolution formed by revolving the region bounded by

\[ y = x - x^3 \]

and the x-axis \((0 \leq x \leq 1)\) about the y-axis.

**Solution** Because the axis of revolution is vertical, use a vertical representative rectangle, as shown in Figure 7.30. The width \(\Delta x\) indicates that \(x\) is the variable of integration. The distance from the center of the rectangle to the axis of revolution is \(h(x) = x - x^3\).

Because \(x\) ranges from 0 to 1, the volume of the solid is

\[
V = 2\pi \int_a^b p(x)h(x) \, dx = 2\pi \int_0^1 x(x - x^3) \, dx
\]

Apply shell method.

\[
= 2\pi \int_0^1 (-x^4 + x^3) \, dx
\]

Simplify.

\[
= 2\pi \left[ -\frac{x^5}{5} + \frac{x^4}{4} \right]_0^1
\]

Integrate.

\[
= 2\pi \left( -\frac{1}{5} + \frac{1}{3} \right)
\]

\[
= \frac{4\pi}{15}
\]

Try It Exploration A
EXAMPLE 2 Using the Shell Method to Find Volume

Find the volume of the solid of revolution formed by revolving the region bounded by the graph of

\[ x = e^{-y^2} \]

and the \( y \)-axis \((0 \leq y \leq 1)\) about the \( x \)-axis.

Solution

Because the axis of revolution is horizontal, use a horizontal representative rectangle, as shown in Figure 7.31. The width indicates that \( y \) is the variable of integration. The distance from the center of the rectangle to the axis of revolution is \( e^{-y^2} \) and the height of the rectangle is \( e^{-y^2} \). Because \( y \) ranges from 0 to 1, the volume of the solid is

\[
V = 2\pi \int_{0}^{1} y e^{-y^2} \, dy = 2\pi \left[ -\frac{1}{2} e^{-y^2} \right]_{0}^{1} = \pi \left( 1 - \frac{1}{e} \right) \approx 1.986.
\]

NOTE To see the advantage of using the shell method in Example 2, solve the equation \( x = e^{-y^2} \) for \( y \).

\[
y = \begin{cases} 1, & 0 \leq x \leq 1/e \\ \sqrt{-\ln x}, & 1/e < x \leq 1 \end{cases}
\]

Then use this equation to find the volume using the disk method.

Comparison of Disk and Shell Methods

The disk and shell methods can be distinguished as follows. For the disk method, the representative rectangle is always perpendicular to the axis of revolution, whereas for the shell method, the representative rectangle is always parallel to the axis of revolution, as shown in Figure 7.32.
Often, one method is more convenient to use than the other. The following example illustrates a case in which the shell method is preferable.

**EXAMPLE 3  Shell Method Preferable**

Find the volume of the solid formed by revolving the region bounded by the graphs of

\[ y = x^2 + 1, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 1 \]

about the y-axis.

**Solution**  In Example 4 in the preceding section, you saw that the washer method requires two integrals to determine the volume of this solid. See Figure 7.33(a).

\[
V = \pi \int_0^1 (1^2 - 0^2) \, dy + \pi \int_1^2 \left[ 1^2 - \left( \sqrt{y - 1} \right)^2 \right] \, dy
\]

Apply washer method.

\[
= \pi \int_0^1 1 \, dy + \pi \int_1^2 (2 - y) \, dy
\]

Simplify.

\[
= \pi \left[ y \right]_0^1 + \pi \left[ 2y - \frac{y^2}{2} \right]_1^2
\]

Integrate.

\[
= \pi + \pi \left( 4 - 2 - 2 + \frac{1}{2} \right)
\]

\[
= \frac{3\pi}{2}
\]

In Figure 7.33(b), you can see that the shell method requires only one integral to find the volume.

\[
V = 2\pi \int_a^b p(x)h(x) \, dx
\]

Apply shell method.

\[
= 2\pi \int_0^1 x(x^2 + 1) \, dx
\]

\[
= 2\pi \left[ \frac{x^4}{4} + \frac{x^3}{2} \right]_0^1
\]

Integrate.

\[
= 2\pi \left( \frac{3}{4} \right)
\]

\[
= \frac{3\pi}{2}
\]

Suppose the region in Example 3 were revolved about the vertical line \( x = 1 \). Would the resulting solid of revolution have a greater volume or a smaller volume than the solid in Example 3? Without integrating, you should be able to reason that the resulting solid would have a smaller volume because “more” of the revolved region would be closer to the axis of revolution. To confirm this, try solving the following integral, which gives the volume of the solid.

\[
V = 2\pi \int_0^1 (1 - x)(x^2 + 1) \, dx \quad p(x) = 1 - x
\]

**FOR FURTHER INFORMATION**  To learn more about the disk and shell methods, see the article “The Disk and Shell Method” by Charles A. Cable in *The American Mathematical Monthly*. 
**SECTION 7.3 Volume: The Shell Method**

**EXAMPLE 4 Volume of a Pontoon**

A pontoon is to be made in the shape shown in Figure 7.34. The pontoon is designed by rotating the graph of

\[ y = 1 - \frac{x^2}{16}, \quad -4 \leq x \leq 4 \]

about the x-axis, where x and y are measured in feet. Find the volume of the pontoon.

**Solution** Refer to Figure 7.35(a) and use the disk method as follows.

\[ V = \pi \int_{-4}^{4} \left( 1 - \frac{x^2}{16} \right)^2 \, dx \]

Apply disk method.

\[ = \pi \int_{-4}^{4} \left( 1 - \frac{x^2}{8} + \frac{x^4}{256} \right) \, dx \]

Simplify.

\[ = \pi \left[ x - \frac{x^3}{24} + \frac{x^5}{1280} \right]_{-4}^{4} \]

Integrate.

\[ = \frac{64\pi}{15} \approx 13.4 \text{ cubic feet} \]

Try using Figure 7.35(b) to set up the integral for the volume using the shell method. Does the integral seem more complicated?

**Try It**

For the shell method in Example 4, you would have to solve for \( x \) in terms of \( y \) in the equation

\[ y = 1 - \frac{x^2}{16}. \]

Sometimes, solving for \( x \) is very difficult (or even impossible). In such cases you must use a vertical rectangle (of width \( \Delta y \)), thus making \( y \) the variable of integration. The position (horizontal or vertical) of the axis of revolution then determines the method to be used. This is shown in Example 5.

**EXAMPLE 5 Shell Method Necessary**

Find the volume of the solid formed by revolving the region bounded by the graphs of \( y = x^3 + x + 1 \), \( y = 1 \), and \( x = 1 \) about the line \( x = 2 \), as shown in Figure 7.36.

**Solution** In the equation \( y = x^3 + x + 1 \), you cannot easily solve for \( x \) in terms of \( y \). (See Section 3.8 on Newton’s Method.) Therefore, the variable of integration must be \( x \), and you should choose a vertical representative rectangle. Because the rectangle is parallel to the axis of revolution, use the shell method and obtain

\[ V = 2\pi \int_{0}^{1} p(x)h(x) \, dx = 2\pi \int_{0}^{1} (2-x)(x^3 + x + 1 - 1) \, dx \]

Apply shell method.

\[ = 2\pi \int_{0}^{1} (-x^4 + 2x^3 - x^2 + 2x) \, dx \]

Simplify.

\[ = 2\pi \left[ -\frac{x^5}{5} + \frac{x^4}{2} - \frac{x^3}{3} + \frac{x^2}{2} \right]_{0}^{1} \]

Integrate.

\[ = 2\pi \left( -\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + \frac{1}{2} \right) \]

\[ = \frac{29\pi}{15}. \]

**Try It**
Exercises for Section 7.3

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on to view the complete solution of the exercise.

Click on to print an enlarged copy of the graph.

In Exercises 1–12, use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the y-axis.

1. \( y = x \)

2. \( y = 1 - x \)

3. \( y = \sqrt{x} \)

4. \( y = x^2 + 4 \)

5. \( y = x^2, \quad y = 0, \quad x = 2 \)

6. \( y = \frac{1}{2}x^2, \quad y = 0, \quad x = 6 \)

7. \( y = x^2, \quad y = 4x - x^2 \)

8. \( y = 4 - x^2, \quad y = 0 \)

9. \( y = 4x - x^2, \quad x = 0, \quad y = 4 \)

10. \( y = 2x, \quad y = 4, \quad x = 0 \)

11. \( y = \frac{1}{\sqrt{2} \pi} e^{-x^2/2}, \quad y = 0, \quad x = 0, \quad x = 1 \)

12. \( y = \begin{cases} \frac{\sin x}{x}, & x > 0, \\ 1, & x = 0 \end{cases}, \quad y = 0, \quad x = 0, \quad x = \pi \)

In Exercises 13–20, use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the x-axis.

13. \( y = x \)

14. \( y = 2 - x \)

In Exercises 21–24, use the shell method to find the volume of the solid generated by revolving the plane region about the given line.

21. \( y = x^2, \quad y = 4x - x^2, \quad \text{about the line } x = 4 \)

22. \( y = x^2, \quad y = 4x - x^2, \quad \text{about the line } x = 2 \)

23. \( y = 4x - x^2, \quad y = 0, \quad \text{about the line } x = 5 \)

24. \( y = \sqrt{x}, \quad y = 0, \quad x = 4, \quad \text{about the line } x = 6 \)

In Exercises 25 and 26, decide whether it is more convenient to use the disk method or the shell method to find the volume of the solid of revolution. Explain your reasoning. (Do not find the volume.)

25. \((y - 2)^2 = 4 - x\)

26. \(y = 4 - e^x\)

In Exercises 27–30, use the disk or the shell method to find the volume of the solid generated by revolving the region bounded by the graphs of the equations about each given line.

27. \( y = x^2, \quad y = 0, \quad x = 2 \)
   (a) the x-axis   (b) the y-axis   (c) the line \( x = 4 \)

28. \( y = \frac{10}{x^2}, \quad y = 0, \quad x = 1, \quad x = 5 \)
   (a) the x-axis   (b) the y-axis   (c) the line \( y = 10 \)

29. \( x^{1/2} + y^{1/2} = a^{1/2}, \quad x = 0, \quad y = 0 \)
   (a) the x-axis   (b) the y-axis   (c) the line \( x = a \)
30. \( x^{2/3} + y^{2/3} = a^{2/3} \), \( a > 0 \) (hypocycloid)
   (a) the x-axis  (b) the y-axis

**Writing About Concepts**

31. Consider a solid that is generated by revolving a plane region about the y-axis. Describe the position of a representative rectangle when using (a) the shell method and (b) the disk method to find the volume of the solid.

32. The region in the figure is revolved about the indicated axes and line. Order the volumes of the resulting solids from least to greatest. Explain your reasoning.
   (a) x-axis   (b) y-axis   (c) \( x = 5 \)

![Graph of y = x^2]

In Exercises 33 and 34, give a geometric argument that explains why the integrals have equal values.

33. \( \pi \int_{-1}^{1} (x - 1) \, dx = 2 \pi \int_{0}^{2} y [5 - (y^2 + 1)] \, dy \)
34. \( \pi \int_{0}^{4} [16 - (2y)^2] \, dy = 2 \pi \int_{0}^{\pi/2} x \left( \frac{x}{2} \right) \, dx \)

In Exercises 35–38, (a) use a graphing utility to graph the plane region bounded by the graphs of the equations, and (b) use the integration capabilities of the graphing utility to approximate the volume of the solid generated by revolving the region about the y-axis.

35. \( x^{4/3} + y^{4/3} = 1 \), \( x = 0 \), \( y = 0 \), first quadrant
36. \( y = \sqrt{1 - x^2} \), \( y = 0 \), \( x = 0 \)
37. \( y = \frac{3}{2} (x - 2)^3 (x - 6)^2 \), \( y = 0 \), \( x = 2 \), \( x = 6 \)
38. \( y = \frac{2}{1 + e^{1/x}} \), \( y = 0 \), \( x = 1 \), \( x = 3 \)

**Think About It**  In Exercises 39 and 40, determine which value best approximates the volume of the solid generated by revolving the region bounded by the graphs of the equations about the y-axis. (Make your selection on the basis of a sketch of the solid and not by performing any calculations.)

39. \( y = 2e^{-x} \), \( y = 0 \), \( x = 0 \), \( x = 2 \)
   (a) 1/2   (b) 2   (c) 4   (d) 7.5   (e) 15

40. \( y = \tan x \), \( y = 0 \), \( x = 0 \), \( x = \pi/4 \)
   (a) 3.5   (b) 3/2   (c) 8   (d) 10   (e) 1

**41. Machine Part**  A solid is generated by revolving the region bounded by \( y = 2x^2 \) and \( y = 2 \) about the y-axis. A hole, centered along the axis of revolution, is drilled through this solid so that one-fourth of the volume is removed. Find the diameter of the hole.

**42. Machine Part**  A solid is generated by revolving the region bounded by \( y = \sqrt{9 - x^2} \) and \( y = 0 \) about the y-axis. A hole, centered along the axis of revolution, is drilled through this solid so that one-third of the volume is removed. Find the diameter of the hole.

**43. Volume of a Torus**  A torus is formed by revolving the region bounded by the circle \( x^2 + y^2 = 1 \) about the line \( x = 2 \) (see figure). Find the volume of this “doughnut-shaped” solid. (Hint: The integral \( \int_{-1}^{1} \sqrt{1 - x^2} \, dx \) represents the area of a semicircle.)

![Torus Diagram]

**44. Volume of a Torus**  Repeat Exercise 43 for a torus formed by revolving the region bounded by the circle \( x^2 + y^2 = r^2 \) about the line \( x = R \), where \( r < R \).

45. (a) Use differentiation to verify that
   \[ \int x \sin x \, dx = \sin x - x \cos x + C. \]
   (b) Use the result of part (a) to find the volume of the solid generated by revolving each plane region about the y-axis.

(i)

![Graph of y = 2sin x]

(ii)

![Graph of y = 2sin x]

46. (a) Use differentiation to verify that
   \[ \int x \cos x \, dx = \cos x + x \sin x + C. \]
   (b) Use the result of part (a) to find the volume of the solid generated by revolving each plane region about the y-axis.

(i)  
![Graph of y = x^2]

(ii)  
![Graph of y = (x-2)^2]
In Exercises 47–50, the integral represents the volume of a solid of revolution. Identify (a) the plane region that is revolved and (b) the axis of revolution.

47. \(2\pi \int_0^2 x^3 \, dx\)  
48. \(2\pi \int_0^1 y - y^{3/2} \, dy\)  
49. \(2\pi \int_0^6 (y + 2)\sqrt{6 - y} \, dy\)  
50. \(2\pi \int_0^1 (4 - x)e^x \, dx\)

51. **Volume of a Segment of a Sphere** Let a sphere of radius \(r\) be cut by a plane, thereby forming a segment of height \(h\). Show that the volume of this segment is \(\frac{4}{3}\pi h^2(3r - h)\).

52. **Volume of an Ellipsoid** Consider the plane region bounded by the graph of

\[
\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1
\]

where \(a > 0\) and \(b > 0\). Show that the volume of the ellipsoid formed when this region revolves about the \(y\)-axis is \(\frac{4\pi a^2 b}{3}\).

53. **Exploration** Consider the region bounded by the graphs of \(y = ax^n\), \(y = ab^n\), and \(x = 0\) (see figure).

(a) Find the ratio \(R_n(n)\) of the area of the region to the area of the circumscribed rectangle.

(b) Find \(\lim_{n \to \infty} R_n(n)\) and compare the result with the area of the circumscribed rectangle.

(c) Find the volume of the solid of revolution formed by revolving the region about the \(y\)-axis. Find the ratio \(R_n(n)\) of this volume to the volume of the circumscribed right circular cylinder.

(d) Find \(\lim_{n \to \infty} R_n(n)\) and compare the result with the volume of the circumscribed cylinder.

(e) Use the results of parts (b) and (d) to make a conjecture about the shape of the graph of \(y = ax^n\) \((0 \leq x \leq b)\) as \(n \to \infty\).

54. **Think About It** Match each integral with the solid whose volume it represents, and give the dimensions of each solid.

(a) Right circular cone  
(b) Torus  
(c) Sphere  
(d) Right circular cylinder  
(e) Ellipsoid

(i) \(2\pi \int_0^r hx \, dx\)  
(ii) \(2\pi \int_0^r hx \left(1 - \frac{x}{r}\right) \, dx\)

(iii) \(2\pi \int_0^r 2x\sqrt{r^2 - x^2} \, dx\)  
(iv) \(2\pi \int_0^b 2ax \sqrt{1 - \frac{x^2}{b^2}} \, dx\)

(v) \(2\pi \int_{-r}^r (R - x)(2\sqrt{r^2 - x^2}) \, dx\)

55. **Volume of a Storage Shed** A storage shed has a circular base of diameter 80 feet (see figure). Starting at the center, the interior height is measured every 10 feet and recorded in the table.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height</td>
<td>50</td>
<td>45</td>
<td>40</td>
<td>20</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Use Simpson’s Rule to approximate the volume of the shed.

(b) Note that the roof line consists of two line segments. Find the equations of the line segments and use integration to find the volume of the shed.

56. **Modeling Data** A pond is approximately circular, with a diameter of 400 feet (see figure). Starting at the center, the depth of the water is measured every 25 feet and recorded in the table.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>25</th>
<th>50</th>
<th>75</th>
<th>100</th>
<th>125</th>
<th>150</th>
<th>175</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depth</td>
<td>20</td>
<td>19</td>
<td>19</td>
<td>17</td>
<td>15</td>
<td>14</td>
<td>10</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Use Simpson’s Rule to approximate the volume of water in the pond.

(b) Use the regression capabilities of a graphing utility to find a quadratic model for the depths recorded in the table. Use the graphing utility to plot the depths and graph the model.

(c) Use the integration capabilities of a graphing utility and the model in part (b) to approximate the volume of water in the pond.

(d) Use the result of part (c) to approximate the number of gallons of water in the pond if 1 cubic foot of water is approximately 7.48 gallons.
57. Consider the graph of \( y^2 = x(4 - x)^2 \) (see figure). Find the volumes of the solids that are generated when the loop of this graph is revolved around (a) the \( x \)-axis, (b) the \( y \)-axis, and (c) the line \( x = 4 \).

58. Consider the graph of \( y^2 = x^2(x + 5) \) (see figure). Find the volume of the solid that is generated when the loop of this graph is revolved around (a) the \( x \)-axis, (b) the \( y \)-axis, and (c) the line \( x = -5 \).

59. Let \( V_1 \) and \( V_2 \) be the volumes of the solids that result when the plane region bounded by \( y = 1/x, \ y = 0, \ x = 4 \), and \( x = c \ (c > \frac{1}{3}) \) is revolved about the \( x \)-axis and \( y \)-axis, respectively. Find the value of \( c \) for which \( V_1 = V_2 \).