Differential Equations: Growth and Decay

- Use separation of variables to solve a simple differential equation.
- Use exponential functions to model growth and decay in applied problems.

Differential Equations

In the preceding section, you learned to analyze visually the solutions of differential equations using slope fields and to approximate solutions numerically using Euler’s Method. Analytically, you have learned to solve only two types of differential equations—those of the forms

\[ y' = f(x) \quad \text{and} \quad y'' = f(x). \]

In this section, you will learn how to solve a more general type of differential equation. The strategy is to rewrite the equation so that each variable occurs on only one side of the equation. This strategy is called separation of variables. (You will study this strategy in detail in Section 6.3.)

**EXAMPLE 1  Solving a Differential Equation**

Solve the differential equation \( y' = 2x/y \).

**Solution**

\[
\begin{align*}
y' &= \frac{2x}{y} & \text{Write original equation.} \\
yy' &= 2x & \text{Multiply both sides by } y. \\
\int yy' \, dx &= \int 2x \, dx & \text{Integrate with respect to } x. \\
y \, dy &= \int 2x \, dx & dy = y' \, dx \\
\int \frac{1}{2}y^2 &= x^2 + C_1 & \text{Apply Power Rule.} \\
y^2 - 2x^2 &= C & \text{Rewrite, letting } C = 2C_1.
\end{align*}
\]

So, the general solution is given by

\[ y^2 - 2x^2 = C. \]

You can use implicit differentiation to check this result.

**Try It**

In practice, most people prefer to use Leibniz notation and differentials when applying separation of variables. The solution of Example 1 is shown below using this notation.

\[
\begin{align*}
\frac{dy}{dx} &= \frac{2x}{y} \\
y \, dy &= 2x \, dx \\
\int y \, dy &= \int 2x \, dx \\
\frac{1}{2}y^2 &= x^2 + C_1 \\
y^2 - 2x^2 &= C
\end{align*}
\]
Growth and Decay Models

In many applications, the rate of change of a variable is proportional to the value of the variable. If is a function of time , the proportion can be written as shown.

\[
\frac{dy}{dt} = ky
\]

The general solution of this differential equation is given in the following theorem.

**THEOREM 6.1 Exponential Growth and Decay Model**

If is a differentiable function of such that and for some constant , then

\[
y = Ce^{kt}.
\]

 is the initial value of , and is the proportionality constant. Exponential growth occurs when and exponential decay occurs when .

**Proof**

\[
y' = ky
\]

Write original equation.
\[
y \frac{y'}{y} = k
\]

Separate variables.
\[
\int \frac{y'}{y} \, dt = \int k \, dt
\]

Integrate with respect to .
\[
\ln y = kt + C_1
\]

Find antiderivative of each side.
\[
y = e^{kt}e^{C_1}
\]

Solve for .
\[
y = Ce^{kt}
\]

Let .

So, all solutions of are of the form .

Select the Animation button below to see that for an exponential decay model, the rate of change of is proportional to .

**EXAMPLE 2 Using an Exponential Growth Model**

The rate of change of is proportional to . When , ; when , . What is the value of when ?

**Solution** Because , you know that and are related by the equation . You can find the values of the constants and by applying the initial conditions.

\[
2 = Ce^0 \quad \Rightarrow \quad C = 2 \quad \text{When } t = 0, \; y = 2.
\]

\[
4 = 2e^{2k} \quad \Rightarrow \quad k = \frac{1}{2} \ln 2 \approx 0.3466 \quad \text{When } t = 2, \; y = 4.
\]

So, the model is . When , the value of is (see Figure 6.8).
Radioactive decay is measured in terms of half-life—the number of years required for half of the atoms in a sample of radioactive material to decay. The half-lives of some common radioactive isotopes are shown below.

- Uranium (\(^{238}\)U) 4,470,000,000 years
- Plutonium (\(^{239}\)Pu) 24,100 years
- Carbon (\(^{14}\)C) 5715 years
- Radium (\(^{226}\)Ra) 1599 years
- Einsteinium (\(^{254}\)Es) 276 days
- Nobelium (\(^{257}\)No) 25 seconds

**EXAMPLE 3  Radioactive Decay**

Suppose that 10 grams of the plutonium isotope Pu-239 was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram?

**Solution**  Let \( y \) represent the mass (in grams) of the plutonium. Because the rate of decay is proportional to \( y \), you know that

\[
y = Ce^{kt}
\]

where \( t \) is the time in years. To find the values of the constants \( C \) and \( k \), apply the initial conditions. Using the fact that \( y = 10 \) when \( t = 0 \), you can write

\[
10 = Ce^{k(0)} = Ce^0 = C
\]

which implies that \( C = 10 \). Next, using the fact that \( y = 5 \) when \( t = 24,100 \), you can write

\[
5 = 10e^{24,100k}
\]

\[
1/e^{24,100} = e^{-24,100k}
\]

\[
1/2 = e^{-24,100k}
\]

\[
\ln(1/2) = -24,100k = k.
\]

So, the model is

\[
y = 10e^{-0.000028761t}, \quad \text{Half-life model}
\]

To find the time it would take for 10 grams to decay to 1 gram, you can solve for \( t \) in the equation

\[
1 = 10e^{-0.000028761t}
\]

The solution is approximately 80,059 years.

**Try It Exploration A**

From Example 3, notice that in an exponential growth or decay problem, it is easy to solve for \( C \) when you are given the value of \( y \) at \( t = 0 \). The next example demonstrates a procedure for solving for \( C \) and \( k \) when you do not know the value of \( y \) at \( t = 0 \).
EXAMPLE 4  Population Growth

Suppose an experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

Solution  Let be the number of flies at time , where is measured in days. Because when and when you can write

From the first equation, you know that Substituting this value into the second equation produces the following.

So, the exponential growth model is

To solve for reapply the condition when and obtain

So, the original population (when ) consisted of approximately flies, as shown in Figure 6.9.

EXAMPLE 5  Declining Sales

Four months after it stops advertising, a manufacturing company notices that its sales have dropped from 100,000 units per month to 80,000 units per month. If the sales follow an exponential pattern of decline, what will they be after another 2 months?

Solution  Use the exponential decay model where is measured in months.

From the initial condition you know that Moreover, because when you have

So, after 2 more months ( ), you can expect the monthly sales rate to be

See Figure 6.10.
In Examples 2 through 5, you did not actually have to solve the differential equation

\[ y' = ky. \]

(This was done once in the proof of Theorem 6.1.) The next example demonstrates a problem whose solution involves the separation of variables technique. The example concerns Newton’s Law of Cooling, which states that the rate of change in the temperature of an object is proportional to the difference between the object’s temperature and the temperature of the surrounding medium.

**EXAMPLE 6  Newton’s Law of Cooling**

Let \( y \) represent the temperature (in °F) of an object in a room whose temperature is kept at a constant 60°. If the object cools from 100° to 90° in 10 minutes, how much longer will it take for its temperature to decrease to 80°?

**Solution** From Newton’s Law of Cooling, you know that the rate of change in \( y \) is proportional to the difference between \( y \) and 60. This can be written as

\[ y' = k(y - 60), \quad 80 \leq y \leq 100. \]

To solve this differential equation, use separation of variables, as shown.

\[ \frac{dy}{dt} = k(y - 60) \quad \text{Differential equation} \]

\[ \int \frac{1}{y - 60} \, dy = k \int dt \]

\[ \ln|y - 60| = kt + C_1 \quad \text{Integrate each side.} \]

Because \( y > 60 \), \( |y - 60| = y - 60 \), and you can omit the absolute value signs. Using exponential notation, you have

\[ y - 60 = e^{kt+C_1} \quad \Rightarrow \quad y = 60 + Ce^{kt}. \]

Using \( y = 100 \) when \( t = 0 \), you obtain 100 = 60 + Ce^{k(0)} = 60 + C, which implies that \( C = 40 \). Because \( y = 90 \) when \( t = 10 \),

\[ 90 = 60 + 40e^{10k} \]

\[ 30 = 40e^{10k} \]

\[ k = \frac{1}{10} \ln \frac{3}{4} \approx -0.02877. \]

So, the model is

\[ y = 60 + 40e^{-0.02877t} \quad \text{Cooling model} \]

and finally, when \( y = 80 \), you obtain

\[ 80 = 60 + 40e^{-0.02877t} \]

\[ 20 = 40e^{-0.02877t} \]

\[ \frac{1}{2} = e^{-0.02877t} \]

\[ \ln \frac{1}{2} = -0.02877t \]

\[ t \approx 24.09 \text{ minutes.} \]

So, it will require about 14.09 more minutes for the object to cool to a temperature of 80° (see Figure 6.11).
Exercises for Section 6.2

The symbol ♦ indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on S to view the complete solution of the exercise.
Click on M to print an enlarged copy of the graph.

In Exercises 1–10, solve the differential equation.

1. \( \frac{dy}{dx} = x + 2 \)
2. \( \frac{dy}{dx} = 4 - x \)
3. \( \frac{dy}{dx} = y + 2 \)
4. \( \frac{dy}{dx} = 4 - y \)
5. \( y' = \frac{5x}{y} \)
6. \( y' = \frac{\sqrt{x}}{3y} \)
7. \( y' = \sqrt{xy} \)
8. \( y' = x(1 + y) \)
9. \( (1 + x^2)y' - 2xy = 0 \)
10. \( xy + y' = 100x \)

In Exercises 11–14, write and solve the differential equation that models the verbal statement.

11. The rate of change of \( Q \) with respect to \( t \) is inversely proportional to the square of \( t \).
12. The rate of change of \( P \) with respect to \( t \) is proportional to \( 10 - t \).
13. The rate of change of \( N \) with respect to \( s \) is proportional to \( 250 - s \).
14. The rate of change of \( y \) with respect to \( x \) varies jointly as \( x \) and \( L - y \).

Slope Fields In Exercises 15 and 16, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a). To print an enlarged copy of the graph, select the MathGraph button.

15. \( \frac{dy}{dx} = x(6 - y), \quad (0, 0) \)
16. \( \frac{dy}{dx} = xy, \quad (0, \frac{1}{2}) \)

In Exercises 17–20, find the function \( y = f(t) \) passing through the point \( (0, 10) \) with the given first derivative. Use a graphing utility to graph the solution.

17. \( \frac{dy}{dt} = \frac{1}{2} t \)
18. \( \frac{dy}{dt} = -\frac{3}{4} \sqrt{t} \)
19. \( \frac{dy}{dt} = -\frac{1}{2} y \)
20. \( \frac{dy}{dt} = \frac{3}{4} y \)

In Exercises 21–24, write and solve the differential equation that models the verbal statement. Evaluate the solution at the specified value of the independent variable.

21. The rate of change of \( y \) is proportional to \( y \). When \( x = 0, y = 4 \) and when \( x = 3, y = 10 \). What is the value of \( y \) when \( x = 6 \)?
22. The rate of change of \( N \) is proportional to \( N \). When \( t = 0, N = 250 \) and when \( t = 1, N = 400 \). What is the value of \( N \) when \( t = 4 \)?
23. The rate of change of \( V \) is proportional to \( V \). When \( t = 0, V = 20,000 \) and when \( t = 4, V = 12,500 \). What is the value of \( V \) when \( t = 6 \)?
24. The rate of change of \( P \) is proportional to \( P \). When \( t = 0, P = 5000 \) and when \( t = 1, P = 4750 \). What is the value of \( P \) when \( t = 5 \)?

In Exercises 25–28, find the exponential function \( y = Ce^{kt} \) that passes through the two given points.

25. \( y \) \( t \) \( (5, 5) \)
26. \( y \) \( t \) \( (0, 4) \)
27. \( y \) \( t \) \( (5, 5) \)
28. \( y \) \( t \) \( (4, 5) \)

Writing About Concepts

29. Describe what the values of \( C \) and \( k \) represent in the exponential growth and decay model, \( y = Ce^{kt} \).
30. Give the differential equation that models exponential growth and decay.

In Exercises 31 and 32, determine the quadrants in which the solution of the differential equation is an increasing function. Explain. (Do not solve the differential equation.)

31. \( \frac{dy}{dx} = \frac{1}{2} xy \)
32. \( \frac{dy}{dx} = \frac{1}{2} x^2 y \)
Radioactive Decay  In Exercises 33–40, complete the table for the radioactive isotope.

<table>
<thead>
<tr>
<th>Isotope</th>
<th>Half-Life (in years)</th>
<th>Initial Quantity</th>
<th>Amount After 1000 Years</th>
<th>Amount After 10,000 Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>33. 226Ra</td>
<td>1599</td>
<td>10 g</td>
<td></td>
<td></td>
</tr>
<tr>
<td>34. 226Ra</td>
<td>1599</td>
<td>1.5 g</td>
<td></td>
<td></td>
</tr>
<tr>
<td>35. 226Ra</td>
<td>1599</td>
<td>0.5 g</td>
<td></td>
<td></td>
</tr>
<tr>
<td>36. 14C</td>
<td>5715</td>
<td>2 g</td>
<td></td>
<td></td>
</tr>
<tr>
<td>37. 14C</td>
<td>5715</td>
<td>5 g</td>
<td></td>
<td></td>
</tr>
<tr>
<td>38. 14C</td>
<td>5715</td>
<td>3.2 g</td>
<td></td>
<td></td>
</tr>
<tr>
<td>39. 239Pu</td>
<td>24,100</td>
<td>2.1 g</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40. 239Pu</td>
<td>24,100</td>
<td>0.4 g</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

41. Radioactive Decay  Radioactive radium has a half-life of approximately 1599 years. What percent of a given amount remains after 100 years?

42. Carbon Dating  Carbon-14 dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, the amount of 14C absorbed by a tree that grew several centuries ago should be the same as the amount of 14C absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much of the radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal? (The half-life of 14C is 5715 years.)

Compound Interest  In Exercises 43–48, complete the table for a savings account in which interest is compounded continuously.

<table>
<thead>
<tr>
<th>Initial Investment</th>
<th>Annual Rate</th>
<th>Time to Double</th>
<th>Amount After 10 Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>43. $1000</td>
<td>6%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>44. $20,000</td>
<td>5 1/2%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>45. $750</td>
<td></td>
<td>7 1/2 yr</td>
<td></td>
</tr>
<tr>
<td>46. $10,000</td>
<td></td>
<td>5 yr</td>
<td></td>
</tr>
<tr>
<td>47. $500</td>
<td></td>
<td></td>
<td>$1292.85</td>
</tr>
<tr>
<td>48. $2000</td>
<td></td>
<td></td>
<td>$5436.56</td>
</tr>
</tbody>
</table>

Compound Interest  In Exercises 49–52, find the principal P that must be invested at rate r, compounded monthly, so that $500,000 will be available for retirement in t years.

49. r = 7 1/2%, t = 20
50. r = 6%, t = 40
51. r = 8%, t = 35
52. r = 9%, t = 25

Compound Interest  In Exercises 53–56, find the time necessary for $1000 to double if it is invested at a rate of r compounded (a) annually, (b) monthly, (c) daily, and (d) continuously.

53. r = 7%
54. r = 6%
55. r = 8.5%
56. r = 5.5%

Population  In Exercises 57–60, the population (in millions) of a country in 2001 and the expected continuous annual rate of change k of the population for the years 2000 through 2010 are given. (Source: U.S. Census Bureau, International Data Base)

(a) Find the exponential growth model P = Ce^{kt} for the population by letting t = 0 correspond to 2000.
(b) Use the model to predict the population of the country in 2015.
(c) Discuss the relationship between the sign of k and the change in population for the country.

<table>
<thead>
<tr>
<th>Country</th>
<th>2001 Population</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>57. Bulgaria</td>
<td>7.7</td>
<td>-0.009</td>
</tr>
<tr>
<td>58. Cambodia</td>
<td>12.7</td>
<td>0.018</td>
</tr>
<tr>
<td>59. Jordan</td>
<td>5.2</td>
<td>0.026</td>
</tr>
<tr>
<td>60. Lithuania</td>
<td>3.6</td>
<td>-0.002</td>
</tr>
</tbody>
</table>

61. Modeling Data  One hundred bacteria are started in a culture and the number N of bacteria is counted each hour for 5 hours. The results are shown in the table, where t is the time in hours.

<table>
<thead>
<tr>
<th>t</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>126</td>
</tr>
<tr>
<td>2</td>
<td>151</td>
</tr>
<tr>
<td>3</td>
<td>198</td>
</tr>
<tr>
<td>4</td>
<td>243</td>
</tr>
<tr>
<td>5</td>
<td>297</td>
</tr>
</tbody>
</table>

(a) Use the regression capabilities of a graphing utility to find an exponential model for the data.
(b) Use the model to estimate the time required for the population to quadruple in size.

62. Bacteria Growth  The number of bacteria in a culture is increasing according to the law of exponential growth. There are 125 bacteria in the culture after 2 hours and 350 bacteria after 4 hours.

(a) Find the initial population.
(b) Write an exponential growth model for the bacteria population. Let t represent time in hours.
(c) Use the model to determine the number of bacteria after 8 hours.
(d) After how many hours will the bacteria count be 25,000?

63. Learning Curve  The management at a certain factory has found that a worker can produce at most 30 units in a day. The learning curve for the number of units N produced per day after a new employee has worked t days is N = 30(1 - e^{-kt}). After 20 days on the job, a particular worker produces 19 units.

(a) Find the initial population.
(b) Write an exponential growth model for the bacteria population. Let t represent time in hours.
(c) Use the model to determine the number of bacteria after 8 hours.
(d) After how many hours will the bacteria count be 25,000?

64. Learning Curve  If in Exercise 63 management requires a new employee to produce at least 20 units per day after 30 days on the job, find (a) the learning curve that describes this minimum requirement and (b) the number of days before a minimal achiever is producing 25 units per day.
65. **Modeling Data** The table shows the population $P$ (in millions) of the United States from 1960 to 2000. (Source: U.S. Census Bureau)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Population, $P$</td>
<td>181</td>
<td>205</td>
<td>228</td>
<td>250</td>
<td>282</td>
</tr>
</tbody>
</table>

(a) Use the 1960 and 1970 data to find an exponential model $P_1$ for the data. Let $t = 0$ represent 1960.

(b) Use a graphing utility to find an exponential model $P_2$ for the data. Let $t = 0$ represent 1960.

(c) Use a graphing utility to plot the data and graph both models in the same viewing window. Compare the actual data with the predictions. Which model better fits the data?

(d) Estimate when the population will be 320 million.

66. **Modeling Data** The table shows the net receipts and the amounts required to service the national debt (interest on Treasury debt securities) of the United States from 1992 through 2001. The monetary amounts are given in billions of dollars. (Source: U.S. Office of Management and Budget)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Receipts</td>
<td>1091.3</td>
<td>1154.4</td>
<td>1258.6</td>
<td>1351.8</td>
<td>1453.1</td>
</tr>
<tr>
<td>Interest</td>
<td>292.3</td>
<td>292.5</td>
<td>296.3</td>
<td>332.4</td>
<td>343.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Year</th>
<th>1997</th>
<th>1998</th>
<th>1999</th>
<th>2000</th>
<th>2001</th>
</tr>
</thead>
<tbody>
<tr>
<td>Receipts</td>
<td>1579.3</td>
<td>1721.8</td>
<td>1827.5</td>
<td>2025.2</td>
<td>1991.2</td>
</tr>
<tr>
<td>Interest</td>
<td>355.8</td>
<td>363.8</td>
<td>353.5</td>
<td>361.9</td>
<td>359.5</td>
</tr>
</tbody>
</table>

(a) Use the regression capabilities of a graphing utility to find an exponential model $R$ for the receipts and a quartic model $I$ for the amount required to service the debt. Let $t$ represent the time in years, with $t = 0$ corresponding to 1992.

(b) Use a graphing utility to plot the points corresponding to the receipts, and graph the corresponding model. Based on the model, what is the continuous rate of growth of the receipts?

(c) Use a graphing utility to plot the points corresponding to the amount required to service the debt, and graph the quartic model.

(d) Find a function $P(t)$ that approximates the percent of the receipts that is required to service the national debt. Use a graphing utility to graph this function.

67. **Sound Intensity** The level of sound $\beta$ (in decibels), with an intensity of $I$ is

$$\beta(I) = 10 \log_{10} \frac{I}{I_0}$$

where $I_0$ is an intensity of $10^{-16}$ watts per square centimeter, corresponding roughly to the faintest sound that can be heard. Determine $\beta(I)$ for the following.

(a) $I = 10^{-14}$ watts per square centimeter (whisper)

(b) $I = 10^{-9}$ watts per square centimeter (busy street corner)

(c) $I = 10^{-6.5}$ watts per square centimeter (air hammer)

(d) $I = 10^{-4}$ watts per square centimeter (threshold of pain)

68. **Noise Level** With the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Use the function in Exercise 67 to find the percent decrease in the intensity level of the noise as a result of the installation of these materials.

69. **Forestry** The value of a tract of timber is

$$V(t) = 100,000e^{0.8-t}$$

where $t$ is the time in years, with $t = 0$ corresponding to 1998. If money earns interest continuously at 10%, the present value of the timber at any time $t$ is $A(t) = V(t)e^{-0.10t}$. Find the year in which the timber should be harvested to maximize the present value function.

70. **Earthquake Intensity** On the Richter scale, the magnitude $R$ of an earthquake of intensity $I$ is

$$R = \frac{\ln I - \ln I_0}{\ln 10}$$

where $I_0$ is the minimum intensity used for comparison. Assume that $I_0 = 1$.

(a) Find the intensity of the 1906 San Francisco earthquake ($R = 8.3$).

(b) Find the factor by which the intensity is increased if the Richter scale measurement is doubled.

(c) Find $dR/dI$.

71. **Newton’s Law of Cooling** When an object is removed from a furnace and placed in an environment with a constant temperature of 80°F, its core temperature is 1500°F. One hour after it is removed, the core temperature is 1120°F. Find the core temperature 5 hours after the object is removed from the furnace.

72. **Newton’s Law of Cooling** A container of hot liquid is placed in a freezer that is kept at a constant temperature of 20°F. The initial temperature of the liquid is 160°F. After 5 minutes, the liquid’s temperature is 60°F. How much longer will it take for its temperature to decrease to 30°F?

**True or False?** In Exercises 73–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

73. In exponential growth, the rate of growth is constant.

74. In linear growth, the rate of growth is constant.

75. If prices are rising at a rate of 0.5% per month, then they are rising at a rate of 6% per year.

76. The differential equation modeling exponential growth is $dy/dx = ky$, where $k$ is a constant.